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The Geometrical Basis of Crystal Chemistry. VII. On Three-Dimensional Polyhedra and Networks

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The numbers of n-gons which can meet at every point of uniform periodic planar tessellations are limited, *viz.* 6 triangles, 4 quadrilaterals, and 3 hexagons.

An investigation is made of infinite periodic surfaces (three-dimensional polyhedra) which have the property that tessellations $\{n, p\}$ can be inscribed on them such that p is greater than the maximum values possible for planar tessellations. The work is concerned with polyhedra realizable in Euclidean space as opposed to studies of regular tessellations on the hyperbolic plane, and is an extension of earlier studies of three-dimensional networks, of which many new examples are presented together with a classification of such networks.

The five regular (Platonic) solids and the set of thirteen semi-regular (Archimedean) solids derived from them by truncation have been known for a long time. (The set of thirteen semi-regular solids reciprocal to the Archimedean solids were not listed until this was done by Catalan in 1870.) The packing of polyhedra to fill space has been studied by Fedorov (1904) who showed that only five types of polyhedra fill space *when similarly oriented* and by Andreini (1907), who studied the filling of space by semi-regular polyhedra alone or in combination with regular polyhedra. The edges of the polyhedra in such space-fillings form periodic three-dimensional networks, in the more symmetrical of which the same number of edges meets at each point. These networks, the links of which enclose polyhedral cavities, are a special set of the infinite family of three-dimensional networks of which the primitive, body-centred, and all-face-centred cubic lattices of the crystallographer are also members, these being respectively 6-, 8-, and 12-connected networks if each lattice point is connected to its n equidistant neighbours.

An attempt was made some years ago (Wells, 1954, 1955, 1956) to derive systematically some of the simplest periodic 3- and 4-connected three-dimensional networks. Of the 3-connected systems we may distinguish as a special group those in which the shortest circuit starting from any point and returning to the same point is always an n-gon (n^3 or $\{n, 3\}$ nets). Examples of these nets were given in which n has the values 7–10 inclusive. The upper limit of n appears to be 10, though this has not been proved. These networks were described as representing the extension into three dimensions of the series beginning with three of the regular solids, those having 3 edges meeting at each vertex, namely the tetrahedron $\{3, 3\}$, the cube $\{4, 3\}$, and the pentagonal dodecahedron $\{5, 3\}$. The next member is the plane net $\{6, 3\}$, and higher members are periodic three-dimensional 3-connected networks.

In these networks we were concerned with the *edges* of the regular polyhedra. There is obviously a complementary problem relating to the regular solids which are the reciprocals of those already mentioned, this being concerned with the *faces* of the polyhedra. The tetrahedron, octahedron, and icosahedron have respectively 3, 4, and 5 equilateral triangular faces meeting at each vertex. The next member of this series is the plane net $\{3, 6\}$ in which 6 triangles meet at each point. We now enquire what is the nature of the surfaces on which we may draw tessellations having more than six equilateral triangles meeting at each point. On a sphere or Euclidean plane an equilateral triangle is also equiangular, the angle being 60° for the plane triangle and greater than 60° on a sphere. On the hyperbolic plane the equilateral triangle has three equal angles of less than 60° and studies have been made of tessellations $\{3, p\}$ having $p > 6$ on this plane, which is only partly realizable in Euclidean space (see, for example, Coxeter & Moser, 1957). We are concerned here with systems realizable in Euclidean space and shall show that tessellations $\{3, p\}$ with $p > 6$ are possible for triangles which are equilateral in the literal but limited sense of having equal sides but not equal angles, the sides being geodesics on surfaces of varying curvature.

These surfaces are periodic in 2 or 3 dimensions and may be described as infinite polyhedra. They may be derived from 2- or 3-dimensional networks, in which 3 or more links meet at every point, by inflating the links until they become tunnels and then inscribing the tessellation of triangles or other polygons on the surface so formed. (Examples of symmetrical 3-dimensional frameworks are shown in Fig. 1.) In Fig. 12, for example, the basic framework is the diamond network (Fig. 1(c)), and surfaces based on 6-tunnel

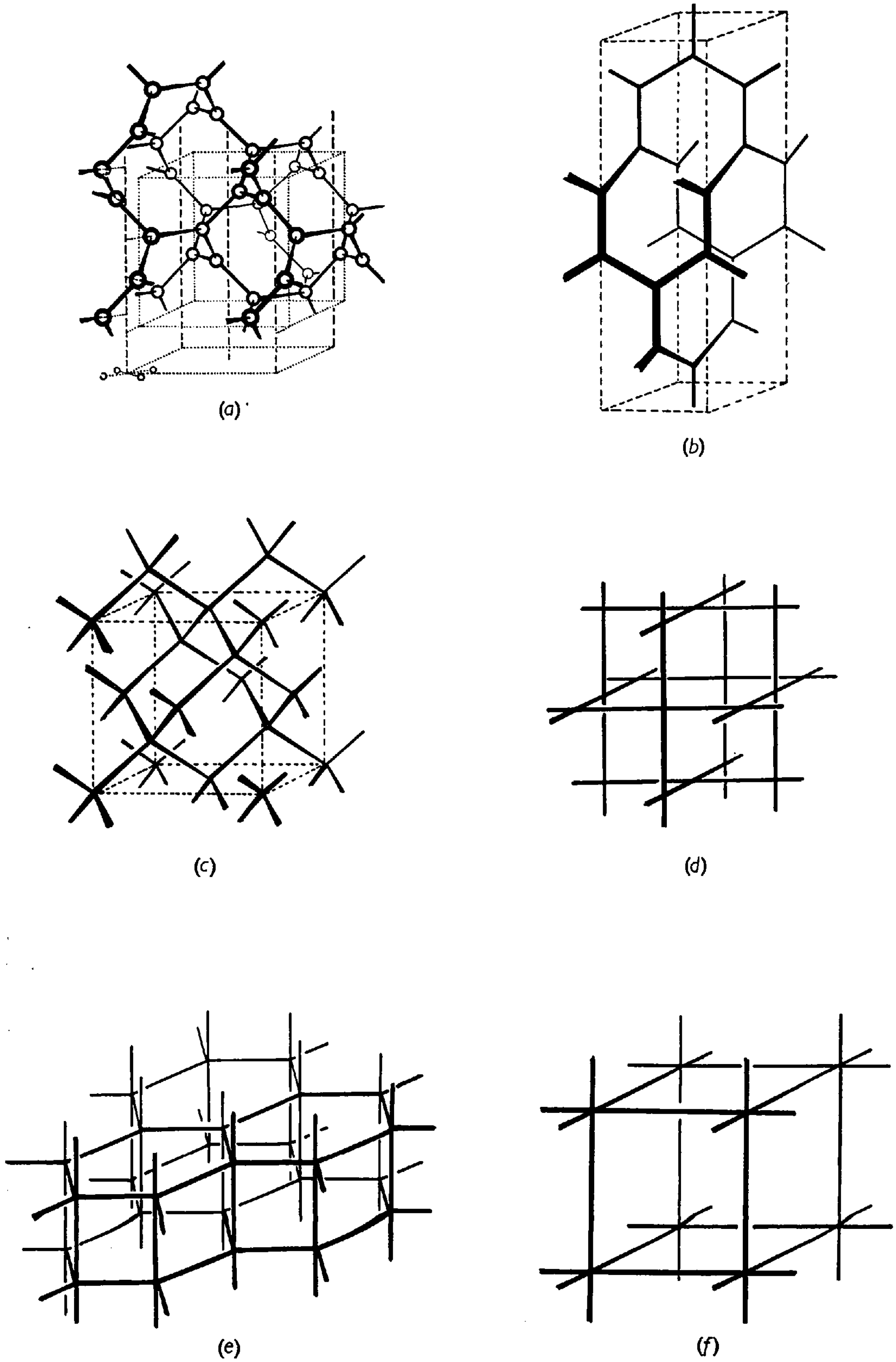

Fig. 1. Three-dimensional networks: (*a*) and (*b*) 3-connected, (*c*) and (*d*), 4-connected, (*e*) 5-connected, and (*f*) 6-connected.

units may be visualized by placing spheres at the points of a primitive lattice (Fig. 1(*f*)) and joining them by tunnels along the cell edges. The tessellations on these 3-dimensional surfaces have the property that the number of n-gons which may meet at every point is greater than for a plane surface.

Table 1

		p							
		3	4	5	6	7	8	9	10
n	3	t	o	i	3,6	three-dimensional triangulated polyhedra			
	4	c	4,4						
	5	d							
	6	6,3							
	7	3-dimensional 3-connected nets							
	8								
	9								
	10								

The triangulated polyhedra form only one group of a much larger family. Thus, passing from the cube {4, 3} through the plane net {4, 4} we come to a family of polyhedra having more than four 4-gons meeting at each point, and similarly for higher polygons. It would appear that we are studying the region below and to the right of the heavy line in Table 1, which includes the regular solids (at the top left-hand corner) and the three regular plane nets, {3, 6}, {4, 4}, and {6, 3}. Here n is the number of sides of the polygons of which p meet at each point. The three-dimensional $\{n, 3\}$ nets occupy the lower part of the first column of Table 1 and the triangulated polyhedra the right-hand part of the top row. In fact we shall see shortly that a diagram of this type is inadequate.

In the case of a triangulated polyhedron the surface is obvious, but as n increases it becomes increasingly difficult to distinguish the surface. It is indeed rather artificial to consider such systems as polyhedra at all; they are best described as networks. For this reason, and also because we wish to include all three-dimensional networks in one scheme, we shall exclude all $\{n, p\}$ in which there are circuits smaller than n-gons, for example, two of the three three-dimensional regular skew polyhedra of Coxeter (1937), namely, his {6, 4/4} and {6, 6/3} ((*B*) and (*C*) in Fig. 2).

Nomenclature

In the Schläfli symbol $\{n, p\}$ for a polyhedron or plane net p may be defined as *either* the number of n-gons *or* the number of links meeting at a point (the 'connectedness' of the net). The two numbers are the same because each edge of a polyhedron is common to two polygons only. In a three-dimensional net a link may be common to more than two n-gons and it becomes necessary to distinguish between the number of polygons and the number of links meeting at a point. We retain p for the latter number.

A three-dimensional net may be more completely described by giving the values of x and/or y, x being the number of n-gons meeting at each point and y the number of n-gons to which each edge is common. (Only in the most symmetrical nets is x the same for all points and y the same for all links.) For polyhedra and plane nets $x=p$ and $y=2$, but for three-dimensional nets x and y may have higher values. For example, for the cubic {10, 3} net of Fig. 1(*a*), $x=15$ and $y=10$, these numbers being related by the relation $p=2x/y$ (Appendix).

We shall begin by considering the nature of the polyhedral surfaces on which p n-gons meet at each point. Since the surfaces join up around tunnels further polygons are formed around the tunnels additional to those we inscribe on the surface. As already noted we exclude cases where these new polygons have fewer than n sides. If further n-gons arise around the tunnels then $y>2$, and in $\{n, p\}$ the number of n-gons meeting at a point is greater than p. If the polygons around the tunnels are all larger than n-gons then $y=2$ as for polyhedra and plane nets, and the number of n-gons meeting at a point is p.

A general classification of three-dimensional networks

It is now evident that we may place all periodic three-dimensional networks on a diagram in which we plot along three axes n, p, and either y or x, as

Fig. 2. The three-dimensional regular skew polyhedra of Coxeter: (*A*) {4, 6/4}, (*B*) {6, 4/4}, and (*C*) {6, 6/3}.

in Fig. 3. In the base are found the regular solids, the regular plane nets, and those three-dimensional nets having $y=2$. Nets with higher values of y lie on higher levels, for example, the cubic $\{10, 3\}$ net. If we choose to give mean values of y for nets in

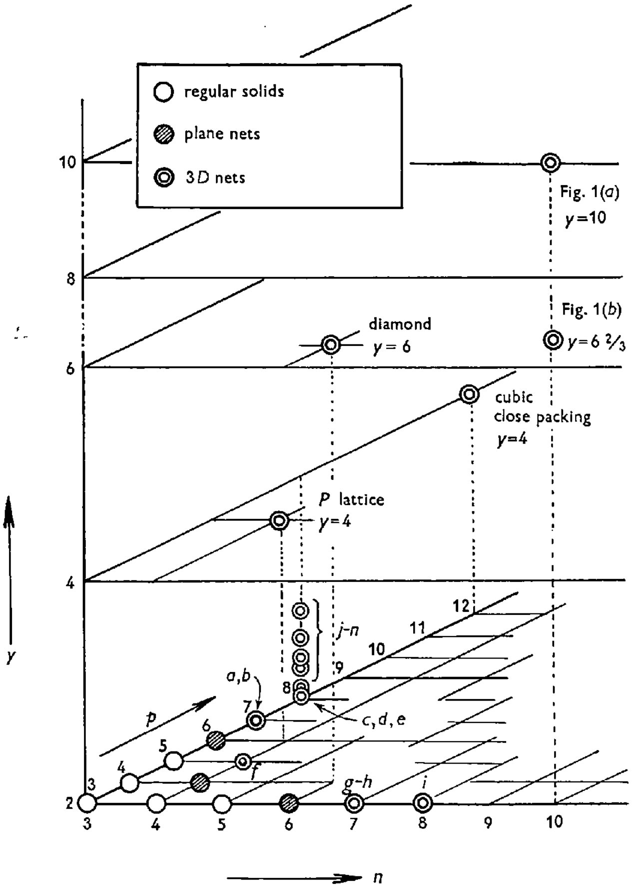

Fig. 3. A classification of nets.
The letters a–n represent the following nets:

a	$\{3, 7\}$	Fig. 7	h	$\{7, 3\}$	reciprocal of b
b	$\{3, 7\}$	Fig. 6(c)	i	$\{8, 3\}$	Fig. 26 (reciprocal of k)
c	$\{3, 8\}$	Fig. 10(a)	j	$\{3, 8\}$	Fig. 16; $y=2\frac{1}{12}$
d	$\{3, 8\}$	Fig. 15	k	$\{3, 8\}$	Fig. 13; $y=2\frac{1}{4}$
e	$\{3, 8\}$	Fig. 17	l	$\{3, 8\}$	Fig. 18; $y=2\frac{1}{3}$
f	$\{4, 5\}$	Fig. 21(d)	m	$\{3, 8\}$	Fig. 12; $y=2\frac{1}{2}$
g	$\{7, 3\}$	reciprocal of a	n	$\{3, 8\}$	Fig. 11; $y=2\frac{3}{4}$

which all the links are not equivalent then such nets may be represented by points between the planes of Fig. 3; they may have non-integral values of y. Examples of such nets include the $\{3, 8\}$ nets j–n of Fig. 3 with values of y between 2 and 3.

Reciprocal pairs of nets

Of the regular solids the tetrahedron is reciprocal to itself, the cube to the octahedron, and the regular dodecahedron to the icosahedron. It has proved useful to derive reciprocals of triangular tessellations since some of these are new 3-connected nets. The links of the reciprocal net connect a point within each triangle to points within the three neighbouring triangles that share edges with the first. In the reciprocal of $\{3, p\}$ the surface tessellation of triangles is replaced by a 3-connected net of p-gons, but there are additional polygons the sizes of which depend on the numbers (c) of triangles in the closed circuits of triangles that are joined through common edges around the tunnels of the original polyhedron $\{3, p\}$. Three cases arise:

$c>p$: reciprocal is a true $\{n, 3\}$ net with no circuits smaller than n-gons, and it has $y=2$ for all links;

$c=p$: reciprocal is a true $\{n, 3\}$ net but with $y>2$ for some or all of the links;

$c<p$: reciprocal has circuits smaller than n-gons and is therefore not a $\{n, 3\}$ net.

Nets of the first group are the closest three-dimensional analogues of the regular solids; examples are included in Fig. 3.

Derivation of three-dimensional polyhedra

Since the surface of one of these polyhedra repeats periodically in three dimensions it is necessary only to consider the nature of the repeating unit, and it is convenient to note here what we mean by 'repeat unit'. In any three-dimensional pattern the true repeat unit is that portion which produces the pattern when repeated, *in the same orientation*, at the points of one of the fourteen Bravais lattices. This (crystallographic) repeat unit must be distinguished from the topological repeat unit. For example, in the plane $\{6, 3\}$ net all points are topologically equivalent and the repeat unit is a single point with three 'half-bonds', as indicated by the heavy lines in Fig. 4(a).

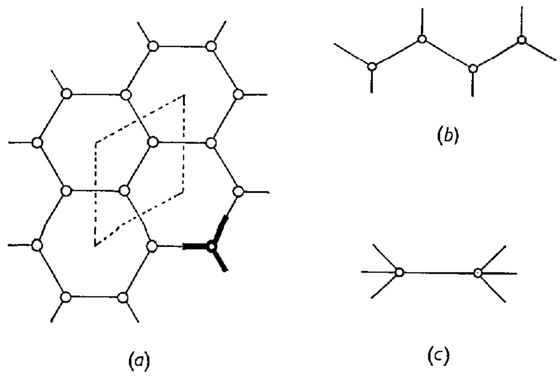

Fig. 4. Repeat units in nets.

The crystallographic repeat unit is, however, the system of two points enclosed within the unit cell (dotted lines). It must have at least four free links to connect with four other units to form an infinite two-dimensional pattern. In a three-dimensional system the crystallographic repeat unit must link up with six other identical units so that, for example, in 3- and 4-connected nets it must consist of a minimum of four 3-connected or two 4-connected points, as shown in Fig. 4(*b*) and (*c*). Therefore if the topological repeat unit is basically a 3-, 4-, or 5-connected unit, as in the 3-, 4-, and 5-tunnel polyhedra described later, the crystallographic repeat unit *must* be a multiple of the topological repeat unit. Even if the topological repeat unit is 6-connected (or more highly connected) the crystallographic repeat unit *may* be larger since it may not be possible to join together the topological repeat units in the same orientation. This is true of the 6-tunnel $\{3, 8\}$ of Fig. 10(*b*). In what follows the term repeat unit normally means the topological repeat unit.

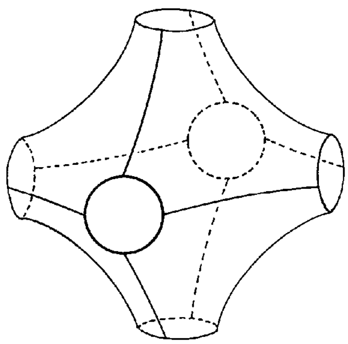

Fig. 5. Derivation of three-dimensional polyhedra (see text).

The number of points Z in the repeat unit of the tessellation $\{n, p\}$ is related to n, p, and the number (t) of tunnels connecting each unit to its neighbours. A (topological) repeat unit may be dissected out of the infinite polyhedron by cutting around each tunnel along links of the tessellation so as to give a polyhedral unit with t faces (holes) representing the tunnels, as shown in Fig. 5 for a 6-tunnel unit. To this finite (simply-connected) polyhedron we may apply Euler's relation: $N_0 - N_1 + N_2 = 2$, where N_0, N_1, and N_2 are the numbers of its vertices, edges and faces. The values of N_0, N_1, and N_2 are all different from the corresponding quantities for the repeat unit of the infinite polyhedral surface, the values for which may be written Z, E, and F. Clearly, $N_2 = F + t$. Since the t faces are shared when the units are joined together, the corners and edges of these faces count as only half-points or half-edges for the repeat unit but as whole points or edges for the finite polyhedron. However, a polygon has the same number of edges as corners and therefore the number of shared points is the same as the number of shared edges. The excess of edges over vertices therefore remains the same for both the connected and unconnected units, so that $Z - E = N_0 - N_1$. Hence

$$Z - E + F + t = 2, \quad \text{or} \quad Z - E + F = 2 - t\,.$$

Since p n-gons meet at every point of the tessellation and every n-gon has n vertices, the ratio of n-gons to points is p/n, or $F = Zp/n$. Since p edges meet at every point and each edge connects two points, the ratio of edges to points is $p/2$, or $E = Zp/2$. Substitution in the modified Euler equation gives

$$Z = \frac{n(2t-4)}{(n-2)p-2n}\,.$$

Table 2. *Polyhedra with 6-tunnel repeat units*

n	Z	p = 3	4	5	6	7	8	9	10
3	$24/(p-6)$	-8	-12	-24	∞	24	12	8	6
4	$32/(2p-8)$	-16	∞	16	8	—	4	—	
5	$40/(3p-10)$	-40	20	8	5				
6	$48/(4p-12)$	∞	12	6					
7	$56/(5p-14)$	56	—						
8	$64/(6p-16)$	32	8						
9	$72/(7p-18)$	24							
10	$80/(8p-20)$	20							

For any specified number of tunnels, for example $t = 6$ for polyhedra based on the primitive lattice, we may draw up a table showing the values of Z for tessellations having p n-gons meeting at each point (Table 2). In this paper we include only solutions having positive, finite, values of $Z \geqslant t$. If $Z < t$ the polyhedron of Fig. 5 has fewer corners than faces and all the faces correspond to tunnels. The only example of an infinite polyhedron with $Z < t$ that we have found, a $\{3, 12\}$ based on the octahedron, will be illustrated in Part VIII. (The infinite values correspond to plane nets and the negative values in this particular table are twice the numbers of vertices of the Platonic solids.) Values of Z lying to the right of the heavy line are less than 6 and/or non-integral with the exception of those for $\{6, 5\}$ and $\{8, 4\}$. These solutions are excluded because their reciprocals have $Z < t$, for a polyhedron is not realizable if its reciprocal on the same surface cannot exist. For the same reason the table is terminated at $n = 10$ for $p = 3$. For a given type of tessellation $\{n, p\}$ the value of Z for a unit with $2t - 2$ tunnels is twice that for a unit with t tunnels because two adjacent t-tunnel units may be regarded as one $(2t-2)$-tunnel unit.

The present paper is essentially descriptive. We shall give examples of polyhedra with $t = 3$, 4, 5, 6, 8, and 12; we have derived no examples of surfaces

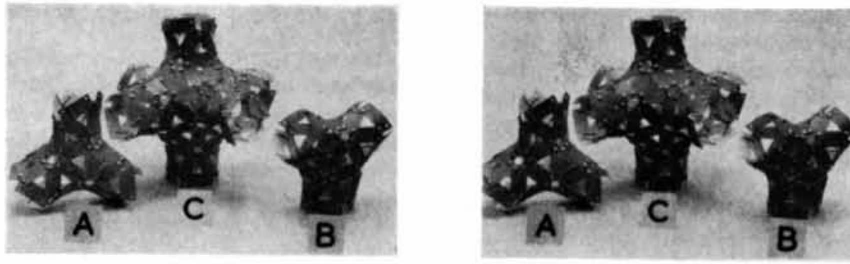

Fig. 6. Repeat units for {3, 7} polyhedra.

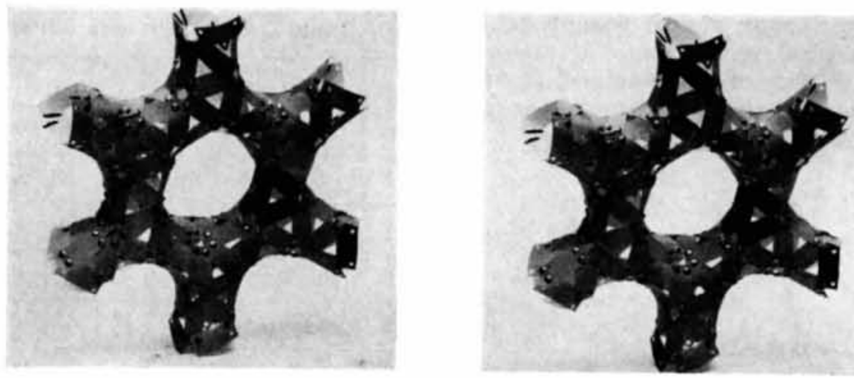

Fig. 7. A planar {3, 7} polyhedron.

with t exceeding 12, which corresponds to the coordination number for closest packing of equal spheres. We do not give further tables of the type of Table 2 because it is more convenient to deal with these polyhedra as families $\{n, p\}$. It is hoped to comment on the topology of these surfaces in a later paper.

Note that all the polyhedra of Table 2 and analogous tables are not necessarily realizable as networks having no polygons smaller than n-gons, since this condition was not introduced in deriving the formula for Z.

Examples of infinite periodic three-dimensional polyhedra

For each family $\{n, p\}$ we give a table showing the values of Z for the different values of t. In these tables * indicates that there is no solution having $Z > t$ and † indicates that the value of Z is non-integral. A number of the polyhedra are illustrated and the figures show either a repeat unit or a larger portion of the infinite periodic polyhedron. In order to accentuate the surfaces the models for n up to 6 are constructed of strips of card linked by paper fasteners. The edges of the n-gons would be the median lines of the strips and the rings of p paper fasteners represent the p-connected points. We deal systematically only with families having n from 3 to 6, and most thoroughly with {3, 8}, but a few examples with higher values of n are illustrated in the form of wire models as networks reciprocal to certain of the triangulated polyhedra. Being built with *straight*

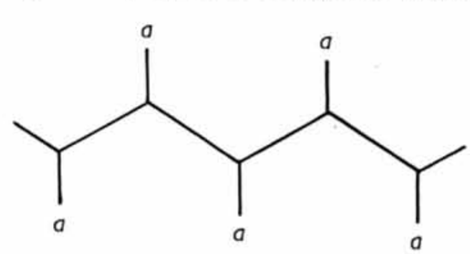

Fig. 8. Portion of the {10, 3} net of Fig. 1(*b*).

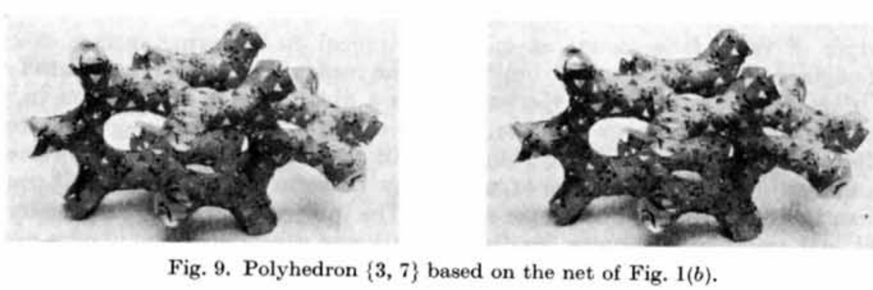

Fig. 9. Polyhedron {3, 7} based on the net of Fig. 1(*b*).

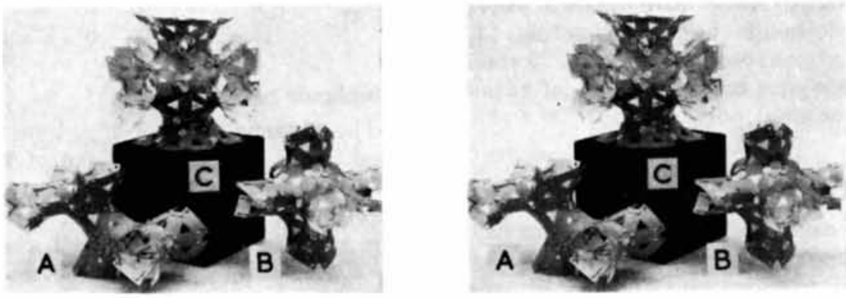

Fig. 10.

Fig. 11.

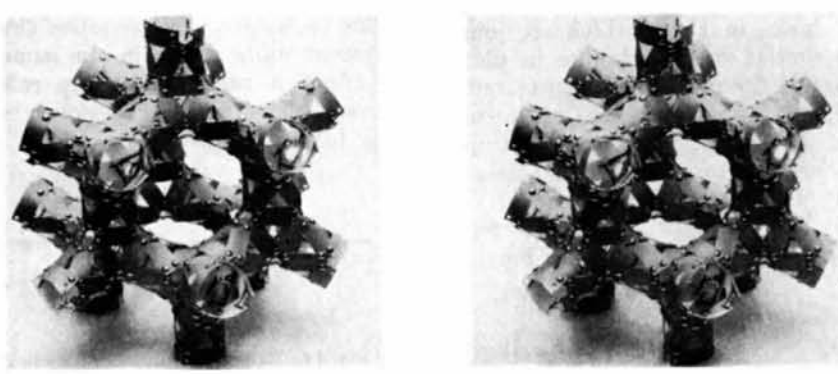

Fig. 12.

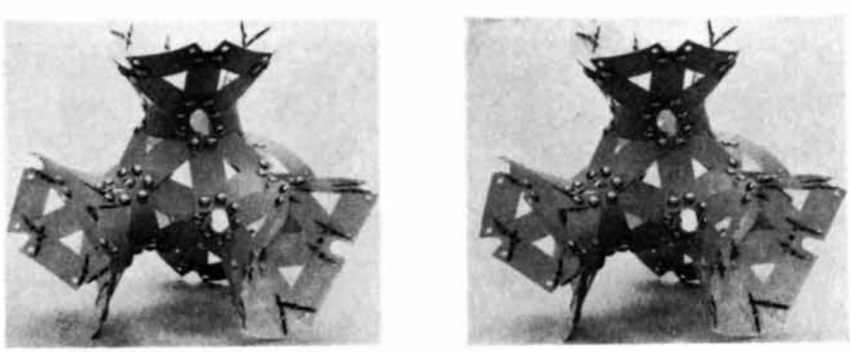

Fig. 13.

Figs. 10–13. Portions of polyhedra {3, 8} (see text).

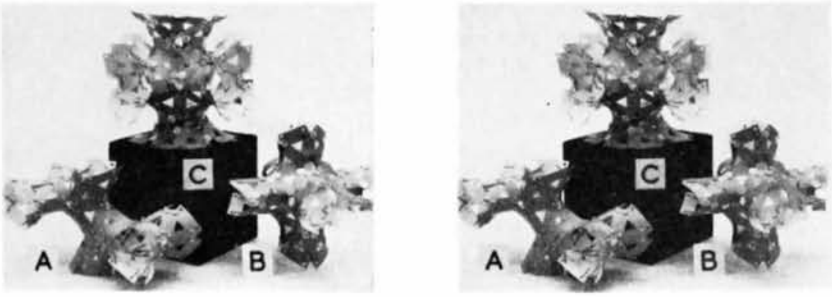

Fig. 10.

Fig. 11.

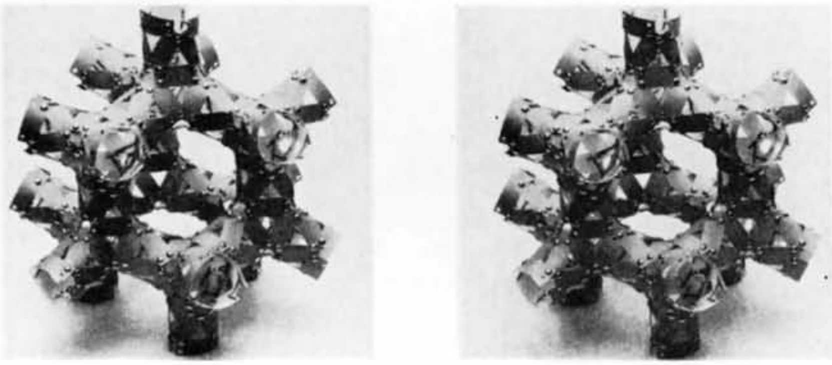

Fig. 12.

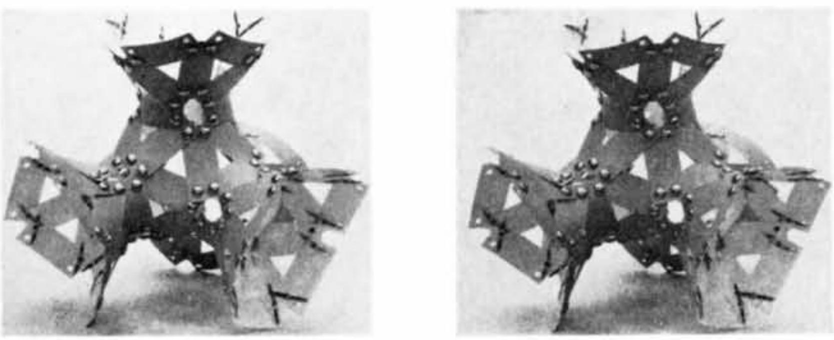

Fig. 13.

Figs. 10–13. Portions of polyhedra {3, 8} (see text).

links (of equal length) these wire models are not strictly the reciprocals of the corresponding $\{3, p\}$ polyhedra, for the true reciprocals (*e.g.* Figs. 25 and 26) are tessellations of p-gons on the surface (of variable curvature) of the original polyhedron.

The $\{3, p\}$ family

This represents the continuation into three dimensions of the series starting with $\{3, 3\}$, tetrahedron, $\{3, 4\}$, octahedron, $\{3, 5\}$, icosahedron, and $\{3, 6\}$, plane triangular net.

$\{3, 7\}$	t	3	4	5	6	8	12
	Z	6	12	18	24	36	60

Examples will be given of the 3- and 6-tunnel polyhedra.

3 coplanar tunnels ($Z=6$)

The unit of Fig. 6(a) is enantiomorphic. Units of the same kind (d- or l-) join up to form the plane net $\{6, 3\}$ (Fig. 7), the simplest two-dimensional 3-connected net. (In the unit cell of the polyhedron there are 12 points, since there are two points in the unit cell of $\{6, 3\}$.) By combining d- and l-units three-dimensional systems arise, of which the simplest is based on the $\{10, 3\}$ net of Fig. 1(b). This arises from rows of d-units as shown in Fig. 8 which are joined at the points a to similar rows of l-units in planes perpendicular to that of the paper. It is illustrated in Fig. 9. In a net built of d- and l-units the repeat unit is the combination (d-+l-), that is, it is a 4-tunnel unit having in the present case $Z=12$. In this connexion see also the $\{4, 7\}$ 8-tunnel polyhedron.

The second $\{3, 7\}$ 3-tunnel unit of Fig. 6(b) forms a rather distorted version of the $\{10, 3\}$ net of Fig. 1(a).

6 octahedral tunnels ($Z=24$)

The 6-tunnel unit of Fig. 6(c) joins up directly if placed at the points of a primitive lattice. A small twist is required at each tunnel if the unit is constructed with equilateral triangles.

$\{3, 8\}$	t	3	4	5	6	8	12
	Z	3	6	9	12	18	30

3 coplanar tunnels ($Z=3$)

The 3-tunnel unit of Fig. 10(a) builds the polyhedron based on the $\{10, 3\}$ net of Fig. 1(b). This polyhedron has $y=2$. Another 3-tunnel unit gives the polyhedron of Fig. 11 based on the $\{10, 3\}$ net of Fig. 1(a). In this polyhedron there are links with $y=2$, 3, and 4, with weighted mean $2\frac{3}{4}$. One ring of the reciprocal $\{8, 3\}$ polyhedron is illustrated in Fig. 25.

4 tetrahedral tunnels ($Z=6$)

Two tetrahedral units have been constructed. The first builds the polyhedron of Fig. 12 based on the diamond net. There are 3-gons around the tunnels (in addition to those on the surface) so that for one-half of the links $y=3$; for the remainder $y=2$, mean $y=2\frac{1}{2}$. The reciprocal has 6-gon in addition to 8-gon circuits and is therefore not an $\{8, 3\}$ net. The second unit (Fig. 13) also builds, with some distortion, a diamond-type polyhedron, for which y (mean)$=2\frac{1}{4}$. A portion of the reciprocal $\{8, 3\}$ net is illustrated in Fig. 26.

4 coplanar tunnels ($Z=6$)

The simplest possibility here is the formation of the plane $\{4, 4\}$ net. This requires that the tunnels from adjacent units all lie in the same plane (Fig. 14(a)). If adjacent repeat units are related by a rotation through 90° then every point is surrounded as in Fig. 14(b).

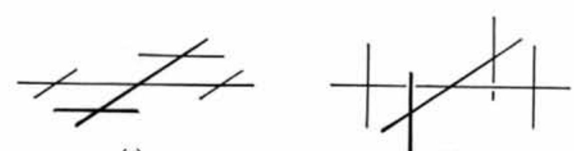

Fig. 14. Relation of adjacent units in 4-tunnel systems.

The three-dimensional polyhedron has the form of the 4-connected net of Fig. 1(d). This (body-centred)

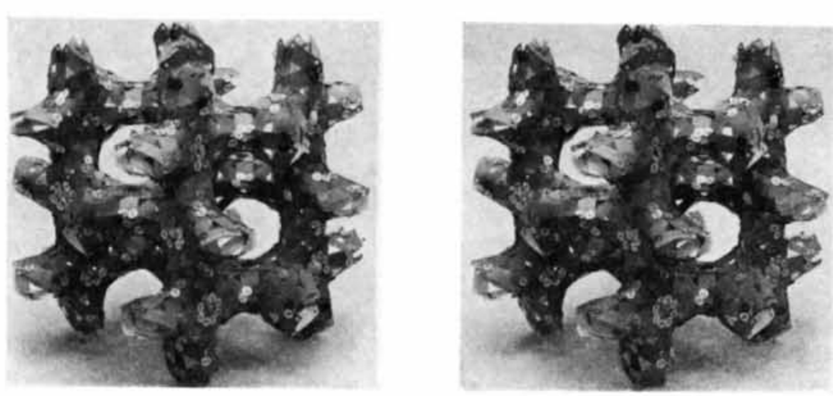

Fig. 15. Polyhedron $\{3, 8\}$ based on the net of Fig. 1(d).

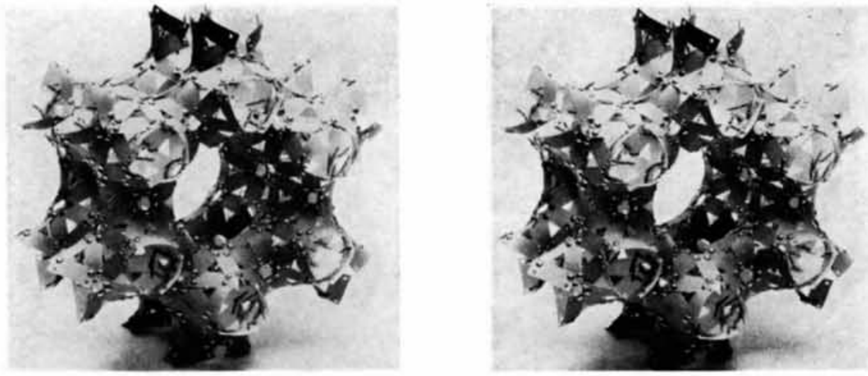

Fig. 16. Polyhedron {3, 8} based on the net of Fig. 1(*e*).

net has 6 points in the unit cell so that the unit cell of the polyhedron (Fig. 15) contains 36 8-connected points. Around the tunnels the shortest circuits of points form 4-gons, *i.e.* $y=2$. The shortest circuit of triangles around the tunnels is one of 8 triangles. The reciprocal (Fig. 27) is therefore a true {8, 3} net having no circuits smaller than 8-gons.

5 *tunnels* ($Z=9$)

The five tunnels are not equivalent, there being three in one plane which link the units into a {6, 3} net and two perpendicular to this plane linking together the sheets, as in Fig. 1(*e*). The polyhedron is illustrated in Fig. 16. For all except 3 of the 36 links in the repeat unit $y=2$ (for the others $y=3$), so that the mean $y=2\frac{1}{12}$.

6 *tunnels* ($Z=12$)

The unit of Fig. 10(*b*), in which all six tunnels are similarly constructed, can be placed at the points of the primitive cubic lattice and forms a polyhedron closely related to the packing of truncated octahedra

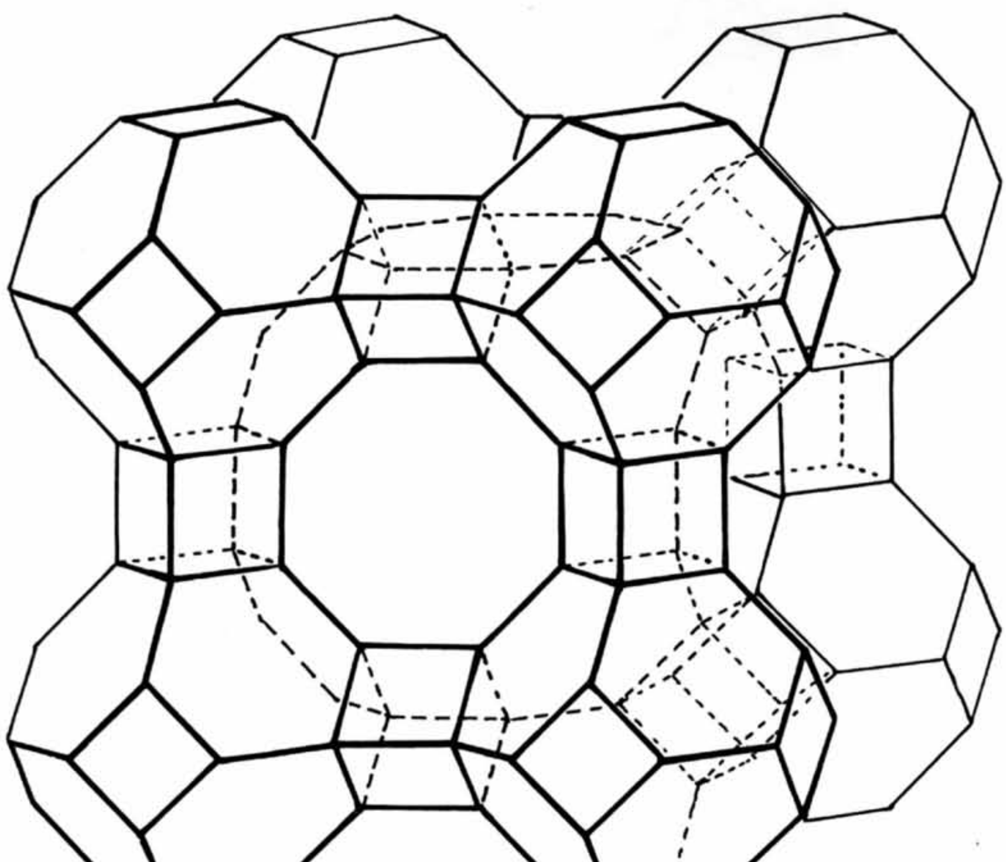

Fig. 17. Open packing of truncated octahedra and cubes.

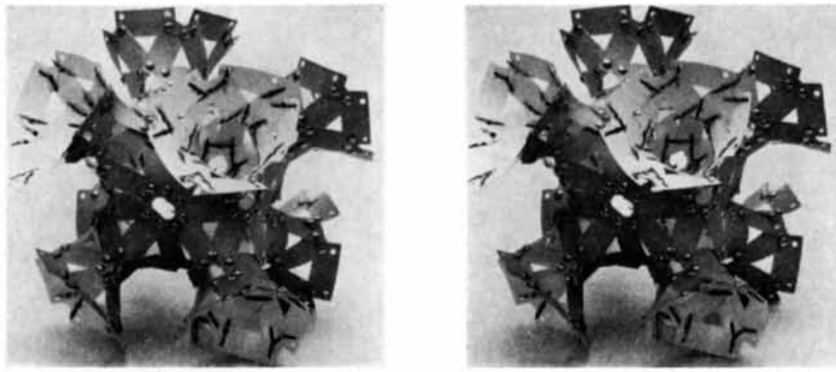

Fig. 18. 8-tunnel repeat unit {3, 8}.

Fig. 19. 12-tunnel repeat unit {3, 8}.

and cubes shown in Fig. 17. For this polyhedron $y=2$. The reciprocal is illustrated in Fig. 28.

The second 6-tunnel unit (Fig. 10(c)) has only tetragonal symmetry, the two polar tunnels being larger than the four equatorial ones. The reciprocal {8, 3} net is shown in Fig. 29.

8 *tunnels* ($Z=18$)

The 8-tunnel unit of Fig. 18 links up to form a polyhedron based on the body-centred cubic lattice. Of the 72 links in the repeat unit 52 have $y=2$, 16 have $y=3$, and 4 have $y=4$, so that the mean $y=2\frac{1}{2}$. The reciprocal {8, 3} is illustrated in Fig. 30.

12 *tunnels* ($Z=30$)

A 12-tunnel unit is illustrated in Fig. 19.

{3, 9}	t	3	4	5	6	8	12
	Z	*	4	6	8	12	20

A 6-tunnel unit is illustrated in Fig. 20(a). The reciprocal {9, 3} net is illustrated in Fig. 31.

{3, 10}	t	3	4	5	6	8	12
	Z	*†	*	*†	6	9	15

The 6-tunnel unit is illustrated in Fig. 20(b).

Calculation of Z for systems {3, 11} shows that $Z=12$ for $t=12$. Although it seems unlikely that this system could be realized we note it here since we include all polyhedra with $Z \gtrless t$. For all polyhedra {3, 12} $Z=t-2$. We have already referred to one of this type as the only example we have found of a polyhedron having $Z<t$.

The {4, *p*} family

This family starts with the cube, {4, 3}, which is followed by the plane net {4, 4}. All higher members are polyhedra based on 2- or 3-dimensional nets.

{4, 5}	t	3	4	5	6	8	12
	Z	4	8	12	16	24	40

The 4-tunnel unit, with $Z=8$, of which two are shown in Fig. 21(a), joins up to form either a planar or a three-dimensional system. In the former case the basic net is the simple {4, 4} net, whereas in the latter case adjacent units are turned through 90° and the basic net is the 4-connected net of Fig. 1(d). Fig. 21(b) shows a second 4-tunnel unit.

Fig. 20.

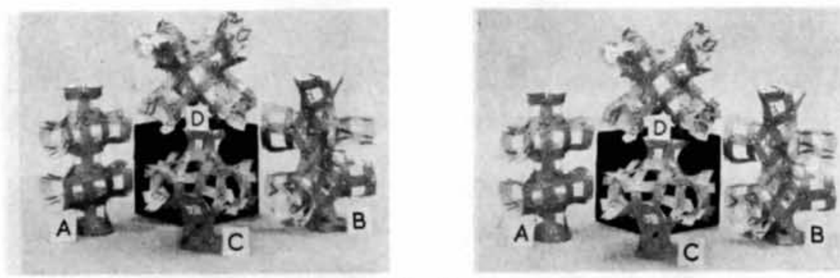

Fig. 21.

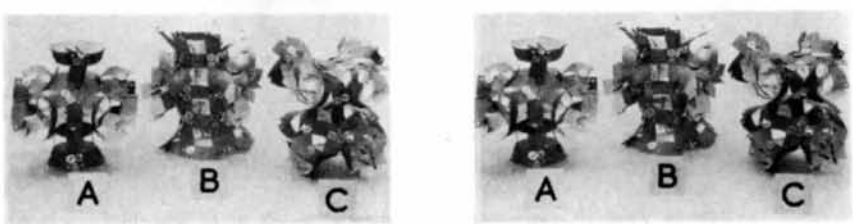

Fig. 22.

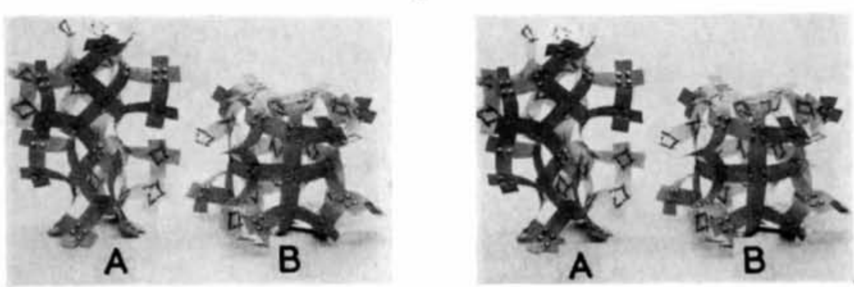

Fig. 23.

Figs. 20–23. Repeat units of polyhedra (see text).

6-tunnel and 8-tunnel units are illustrated in Fig. 21(c) and (d).

$\{4, 6\}$	t	3	4	5	6	8	12
	Z	*	4	6	8	12	20

It is interesting that the values of Z for this group are the numbers of vertices of the regular solids, as in the case of $\{3, 9\}$. Two 6-tunnel units are illustrated in Fig. 22; the links of Fig. 22(a), which is Coxeter's $\{4, 6/4\}$, correspond to the primitive lattice.

$\{4, 7\}$	t	3	4	5	6	8	12
	Z	*†	*†	†	*†	8	†

There is only one solution in this, the last member of the $\{4, p\}$ family. Fig. 22(c) illustrates one form of the 8-tunnel unit. The top half is the mirror image of the lower half. (If a polyhedron were built of units of which the two halves were both left- or both right-handed the unit would then be a 5-tunnel unit which would have $Z=4$ ($i.e.$ $Z<t$)).

In $\{4, 8\}$ $Z=t-2$ throughout, so that none is realizable with $Z \geqslant t$.

The $\{5, p\}$ family

The first member is the pentagonal dodecahedron, $\{5, 3\}$. There is no plane net with an integral value of p but there is the mixed net $\{5, {}^{3}_{4}\}$. The members from $\{5, 4\}$ onwards are three-dimensional polyhedra. There are only two classes in this family, $\{5, 4\}$ and $\{5, 5\}$.

$\{5, 4\}$	t	3	4	5	6	8	12
	Z	5	10	15	20	30	50

Members of this family are 4-connected nets built of

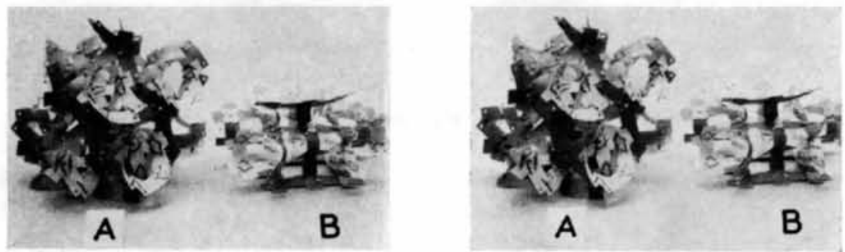

Fig. 24. Repeat units of polyhedra: (*a*) $\{5, 5\}$, (*b*) $\{6, 4\}$.

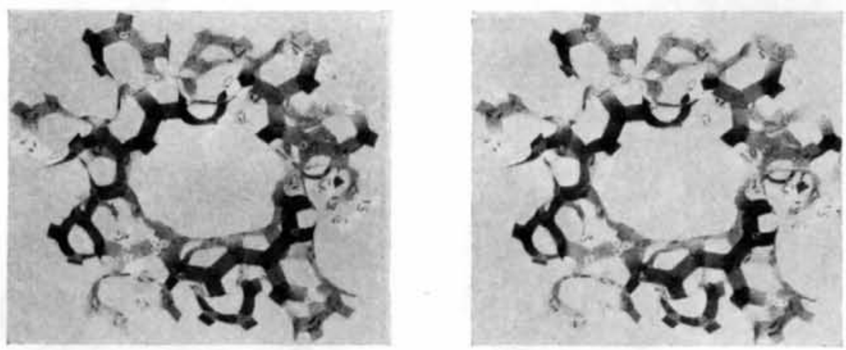

Fig. 25. One 10-ring of $\{8, 3\}$ reciprocal to $\{3, 8\}$ of Fig. 11.

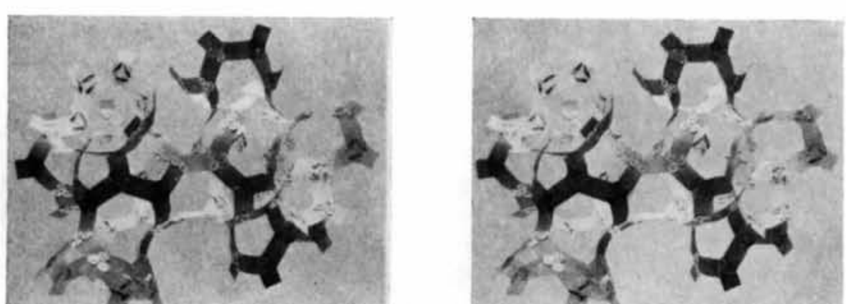

Fig. 26. Portion of $\{8, 3\}$ reciprocal to polyhedron $\{3, 8\}$ built of the unit of Fig. 13.

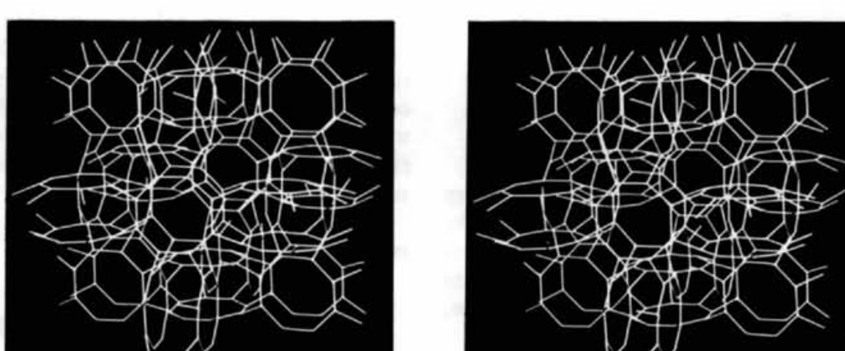

Fig. 27. $\{8, 3\}$ reciprocal to the $\{3, 8\}$ of Fig. 15.

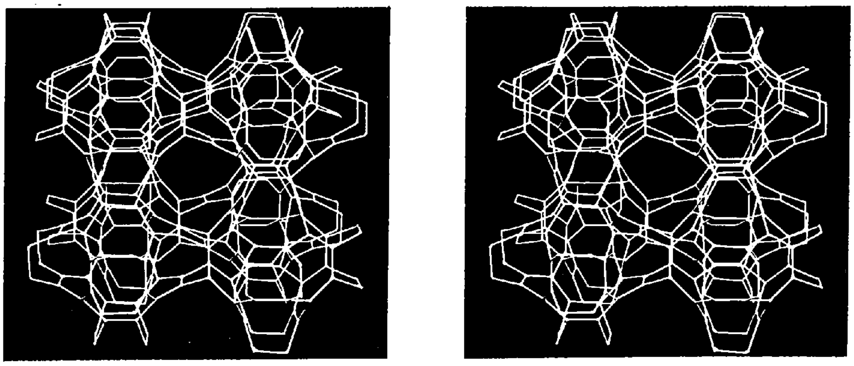

Fig. 28. {8, 3} reciprocal to the polyhedron built of the unit of Fig. 10(*b*).

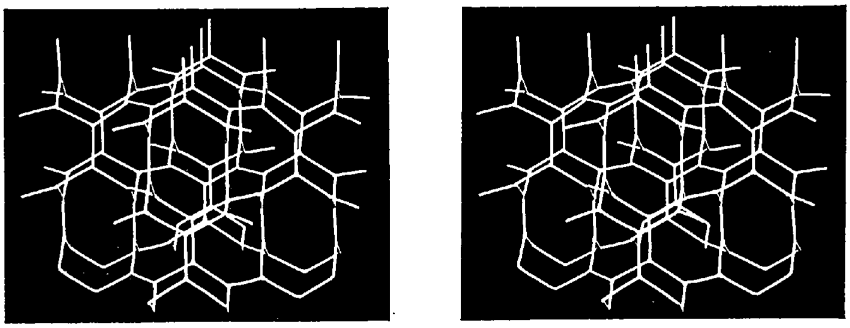

Fig. 29. {8, 3} reciprocal to the polyhedron built of the unit of Fig. 10(*c*).

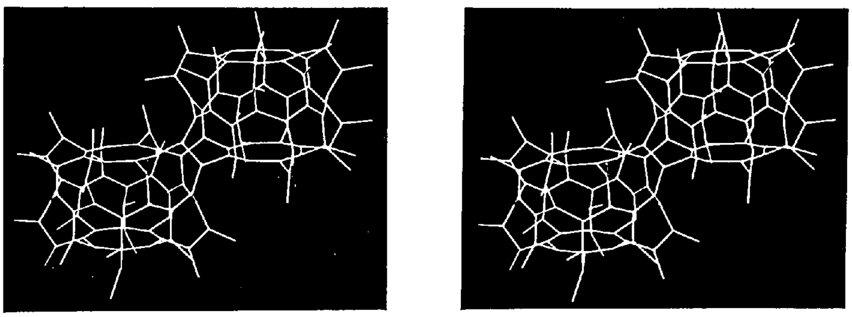

Fig. 30. {8, 3} reciprocal to the polyhedron built of the unit of Fig. 18.

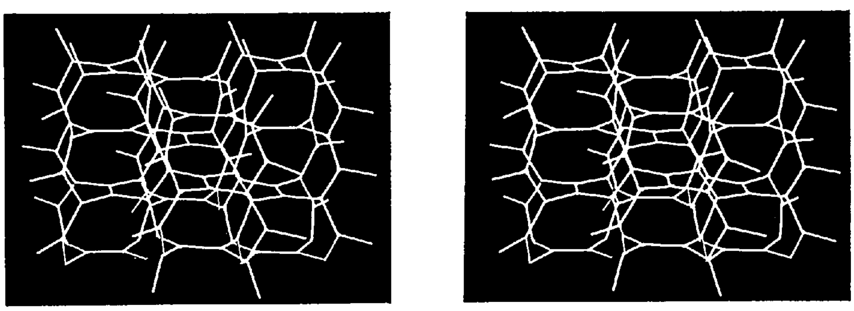

Fig. 31. {9, 3} reciprocal to the polyhedron built of the unit of Fig. 20(*a*).

pentagons, that is, they are intermediate between the plane net {4, 4} and the diamond net {6, 4}. In Fig. 23 we illustrate (*a*) a pair of 4-tunnel units which form the polyhedron based on the net of Fig. 1(*d*), and (*b*) an 8-tunnel unit which forms a polyhedron based on the body-centred cubic lattice. These two units are the reciprocals of the {4, 5} units of Fig. 21(*b*) and (*d*).

{5, 5}	*t*	3	4	5	6	8	12
	Z	*	4	6	8	12	20

This class, with the same numerical values of Z as {3, 9} and {4, 6}, is of special interest as a continuation of the series: {3, 3}, tetrahedron, and {4, 4}, plane net. The reciprocal of any polyhedron in this class is identical with the original polyhedron. The 4-tunnel {5, 5} is illustrated in Fig. 24(*a*).

The {6, *p*} family

This family represents the continuation into three dimensions of the series starting with the plane {6, 3} net. There are two classes only.

{6, 4}	*t*	3	4	5	6	8	12
	Z	3	6	9	12	18	30

The values of Z are the same as for {3, 8}. Fig. 24(*b*) shows the 6-tunnel unit which is the reciprocal of the 6-tunnel {4, 6} of Fig. 22(*b*). Another 6-tunnel unit, the reciprocal of that of Fig. 22(*a*), is the skew polyhedron {6, 4/4} of Coxeter which is not admissible as a {6, 4} net because it contains 4-gon circuits.

{6, 5}	*t*	3	4	5	6	8	12
	Z	*†	*	*	6	9	15

The values of Z are the same as for {3, 10}. It has not yet been possible to find a polyhedron of this class which has no polygons smaller than 6-gons.

Some new 3-connected nets

The reciprocals of triangulated polyhedra are 3-connected nets. We list in Table 3 the values of Z for different values of n and t. Unlike the 3-connected nets derived in earlier papers some of these new nets have $y=2$. These are therefore the closest analogues in three dimensions of the 3-connected regular solids and plane net {6, 3}. A particularly interesting new type of 3-connected net arises from the triangulated polyhedra built from 3-tunnel units. For example, the {3, 8} polyhedron of Fig. 11 is based on the cubic {10, 3} net. The reciprocal {8, 3} net is a 3-connected system of octagons the basic framework of which is the 3-connected 10-gon net. This process is analogous to the replacement of a 3-connected point in a plane net by a triangle. It can presumably not continue further because in the reciprocal {8, 3} net, for example, the points are no longer equivalent.

Table 3. *Possible types of* {*n*, 3} *nets*

Values of Z

n	Number of tunnels (*t*) 3	4	5	6	8	12
7	14	28	42	56	84	140
8	8	16	24	32	48	80
9		12	18	24	36	60
10			15	20	30	50
11						44 ?

APPENDIX

Relation between x, y, and p

In a p-connected net in which the shortest circuits are n-gons and all points and edges are equivalent, let x n-gons meet at each point and let each edge be common to y n-gons. Then in any n-gon each edge counts as $1/y$ so that the total number of edges is $(x/y)m$ if m is the total number of n-gons. Since each point is common to x n-gons the total number of points is $(n/x)m$. In a p-connected net the ratio of edges to point is $p/2$, therefore:

$$p(n/x)=2(n/y), \quad \text{or} \quad p=2x/y\ .$$

For example, in the 3-connected cubic net of Fig. 1(*a*) $x=15$ and $y=10$, whence $2x/y=p=3$.

Now suppose that all the points are equivalent but the links are not all equivalent. In any n-gon let a_1 edges be shared between y_1 n-gons, a_2 edges between y_2 n-gons, and so on. Then the total number of edges becomes $(\Sigma a_1/y_1)m$. The number of points is $(n/x)m$ as before, so that

$$\frac{pn}{x}=2\Sigma(a_1/y_1) \quad \text{or} \quad p=\frac{2x}{n}\Sigma\frac{a}{y}\ .$$

For example, in the tetragonal {10, 3} net of Fig. 1(*b*) all points have $x=10$ but the links are of two types

$$a_1=4,\ y_1=8\ ,$$
$$a_2=6,\ y_2=6\ ,$$

and any 10-gon has four links of one kind and six of the other. The value of $\Sigma(a/y)$ is 3/2, consistent with $p=3$.

In the text and in Fig. 3 we give the *weighted mean* value of y for a number of nets having non-equivalent links. If the numbers of the different types of link in the *repeat unit* of the net are q_1 *etc.* then

$$y\ (\text{mean})=\frac{\Sigma q_n y_n}{\Sigma q_n}\ .$$

For the tetragonal net just mentioned $q_1=4$, $y_1=8$, and $q_2=8$, $y_2=6$, whence $y\ (\text{mean})=6\frac{2}{3}$.

We are indebted to Dr I. J. Good for drawing our attention to one of the families of surface tessellations described in this paper. Realizing that only six equilateral triangles can meet at a point on a plane surface he investigated the nature of the (curved) surface which may be divided into equilateral triangles in such a way that eight meet at every point, and in this way derived one of the 6-tunnel {3, 8} polyhedra. Special thanks are due to our colleague Mr E. Young and his staff for preparing the numerous stereoscopic photographs.

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The Crystal and Molecular Structure of (+)-Hetisine Hydrobromide*

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(+)-Hetisine hydrobromide, $C_{20}H_{27}O_3N.HBr$, crystallizes in the monoclinic system,

$$a = 9{\cdot}75,\ b = 10{\cdot}84,\ c = 9{\cdot}46\ \text{Å},\ \beta = 114°40',$$

the space group being $P2_1$ with two molecules per unit cell. The structure was solved from a three-dimensional Fourier synthesis based on phases of the contribution of the bromine atom. The determination of the molecular structure was carried out solely on the empirical formulae, except that the nitrogen atom was identified with the help of chemical work on this alkaloid. The absolute configuration was investigated by reference to the anomalous dispersion of the Cu $K\alpha$ radiation by the Br atom.

Introduction

Hetisine, $C_{20}H_{27}O_3N$, was isolated from *Aconitum heterophyllum* by Jacobs & Craig (1942) and its first preliminary structural investigation by chemical methods was undertaken by Jacobs & Huebner (1947). It was followed by an extensive chemical study, which led to the proposal of two structures (Solo & Pelletier, 1959; Wiesner & Valenta, 1958), given in Fig. 1(*a*) and (*b*), neither of which, however, accounted satisfactorily for the accumulated evidence. Hetisine appeared to have an unusual heptacyclic skeleton, capable of a facile and interesting rearrangement. In view of this, and the limited amount available for chemical study, Dr O. E. Edwards suggested an X-ray analysis of the alkaloid.

The structure shown in Fig. 1(*c*) represents the result of this determination. It proved valuable in establishing in detail the stereochemistry of the carbon–nitrogen skeleton and in locating all the substituents.

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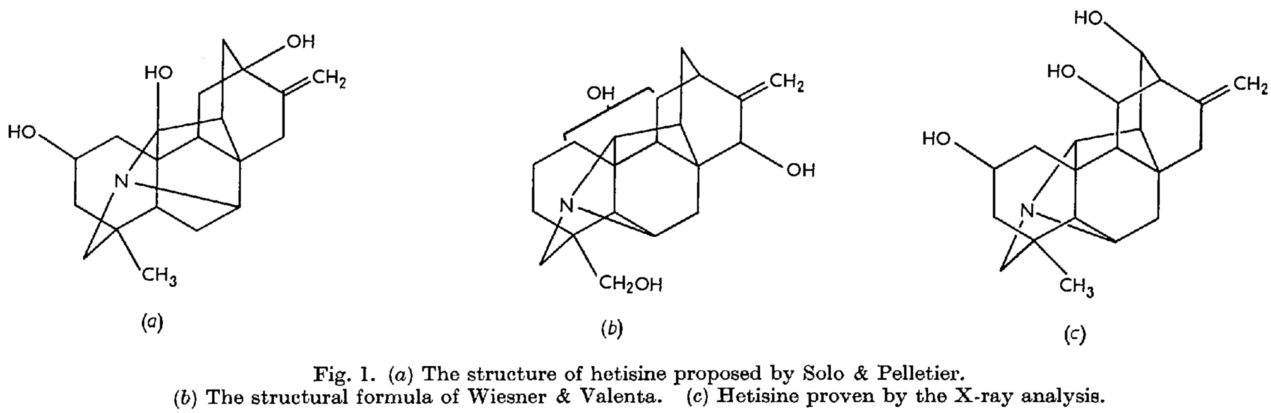

Fig. 1. (*a*) The structure of hetisine proposed by Solo & Pelletier.
(*b*) The structural formula of Wiesner & Valenta. (*c*) Hetisine proven by the X-ray analysis.